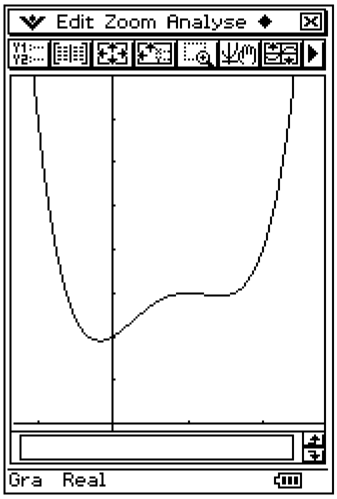
4th degree polynomial and the golden ratio

Die Idee zur Aufgabe stammt von M.Sc. Tor Andersen, einem Lehrer aus Norwegen. Er stellte auf einem internationalen Kongress im Dezember 2007 den Zusammenhang zwischen der Zahl φ des goldenen Schnitts und einem Polynom 4. Grades anhand eines konkreten Beispiels vor und ermunterte die Teilnehmer, diesen Zusammenhang auch für weitere Polynome 4. Grades zu untersuchen.
Nach dem Schriftlichen Abitur 2008 machten sich Natalie Kern, Katja Rösler, Christoph Amann und Benjamin Deck, eine Gruppe von Schülern des Schwarzwald-Gymnasiums Triberg, daran, diesen Zusammenhang allgemein mithilfe des ClassPad nachzuweisen. Um mit Tor Andersen über das Problem kommunizieren zu können, übersetzten die Schülerinnen und Schüler ihre Ausarbeitung in Englische[1].

1. Example with precise numbers:



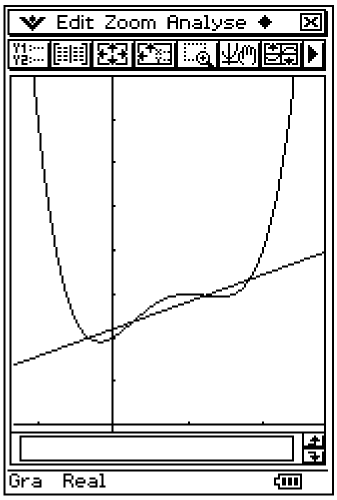
We define an optional function by $f(x) = x^4 - 3x^3 + 2x^2 + x + 2$
and get the following graph:
In order to find the inflection points we compute the first three derivatives. We determine the roots of the second derivative. The values we obtain are inserted into the third derivative. If the result does not equal 0 we have found an inflection point.
In our case, there are two inflection points.

CAS: $\text{solve}(f''(x) = 0, x)$

CAS output: $x = 0.271... \Rightarrow xw1$
$x = 1.228... \Rightarrow xw2$

$f(xw1) = 2.363... \Rightarrow yw1$
$f(xw2) = 2.962... \Rightarrow yw2$

The straight line $y = m \cdot x + b$ through the inflection points intersects the graph of the polynomial in two more points.



In our case, we have:

Definition of m: $m = \dfrac{yw2 - yw1}{xw2 - xw1} = 0.625$

Definition of b: $\text{solve}(yw1 = 0.625 \cdot xw1 + b, b)$

CAS output: $b = 2.194...$

Definition of the straight line $g(x) = 0.625 \cdot x + 2.194$

Points of intersection of the straight line g(x) and the function f(x):

CAS: $\text{solve}(f(x) = g(x), x)$

[1] Die deutsche Version „Polynome 4. Grades und der Goldene Schnitt" wurde im *CASIO forum 2009/1* veröffentlicht.

CAS output:

$$x = xw1$$
$$x = xw2$$
$$x = -0.320$$
$$x = 1.820$$

Now we compute the distance between the points of inflection (d) and the distance between the left intersection point and the left inflection point (a).

Distance between the points of inflection:

$\sqrt{(yw2 - yw1)^2 + (xw2 - xw1)^2} \Rightarrow d$; CAS output: $d = 1.129$

Distance between the left point of intersection and the left inflection point:

$\sqrt{(yw1 - f(-0.320...)^2 + (xw1 - (-0.320)^2} \Rightarrow a$ CAS output: $a = 0.697$

In order to analyse the proportion between the two distances, we divide the distance d by the distance a and vice versa.

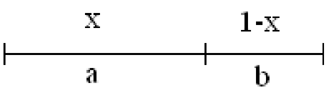
$\frac{d}{a} = 1.618...$ $\frac{a}{d} = 0.618...$

It is noticeable that the results only differ in the numerical digit in front of the comma.

By analysing this fact more precisely we come across the golden ratio.

2. Golden ratio:

A line segment is subdivided into a and b. It should result in the proportion: $\frac{a}{b} = \frac{a+b}{a}$.



It is also possible to use the variable x instead of a.

If the length of the segment is 1, we have $b = 1 - x$

It follows: $\frac{x}{1-x} = \frac{x+(1-x)}{x}$

The “solve” command yields: $x = \frac{\sqrt{5}-1}{2}, x = \frac{-(\sqrt{5}+1)}{2}$

approx(ans):

$$x = 0.618...$$
$$x = -1.618...$$

The first solution is called φ (phi), the absolute value of the second solution is called Φ.

3. General case:

A general polynomial function of degree 4 has the following form:

$$f(x) = a \cdot x^4 + b \cdot x^3 + c \cdot x^2 + d \cdot x + e$$

In order to make things easier, it is useful to remove two parameters.

First of all, the parameter e can be eliminated as it only causes up- and downward shifts, which does not have any effect on the proportion of the line segments introduced in the first part of this paper.

Secondly, we can eliminate the parameter a:

Factoring out a yields: $a \cdot (x^4 + \frac{b}{a} \cdot x^3 + \ldots)$ a can be eliminated and fractions like $\frac{b}{a}$ can be replaced by b,

because a only causes vertical dilation, the proportions remain unchanged.

Consequently, the general function can be simplified to: $f(x) = x^4 + b \cdot x^3 + c \cdot x^2 + d \cdot x$

We use the ClassPad to compute the first two derivatives and find the roots of the second derivative.

The ClassPad gives us two solutions:

$$x = \frac{-(3 \cdot b \pm \sqrt{3 \cdot (3 \cdot b^2 - 8 \cdot c)})}{12}$$

These solutions are stored as xw1 and xw2. Their outputs f(xw1) and f(xw2) are stored as yw1 and yw2.

Straight line $y = m \cdot x + b$ through the two inflection points:

Applying the “simplify” command on the ClassPad, we obtain: $m = \frac{b^3}{8} - \frac{b \cdot c}{2} + d$

$solve(yw2 = m \cdot xw1 + z, z)$ then yields:

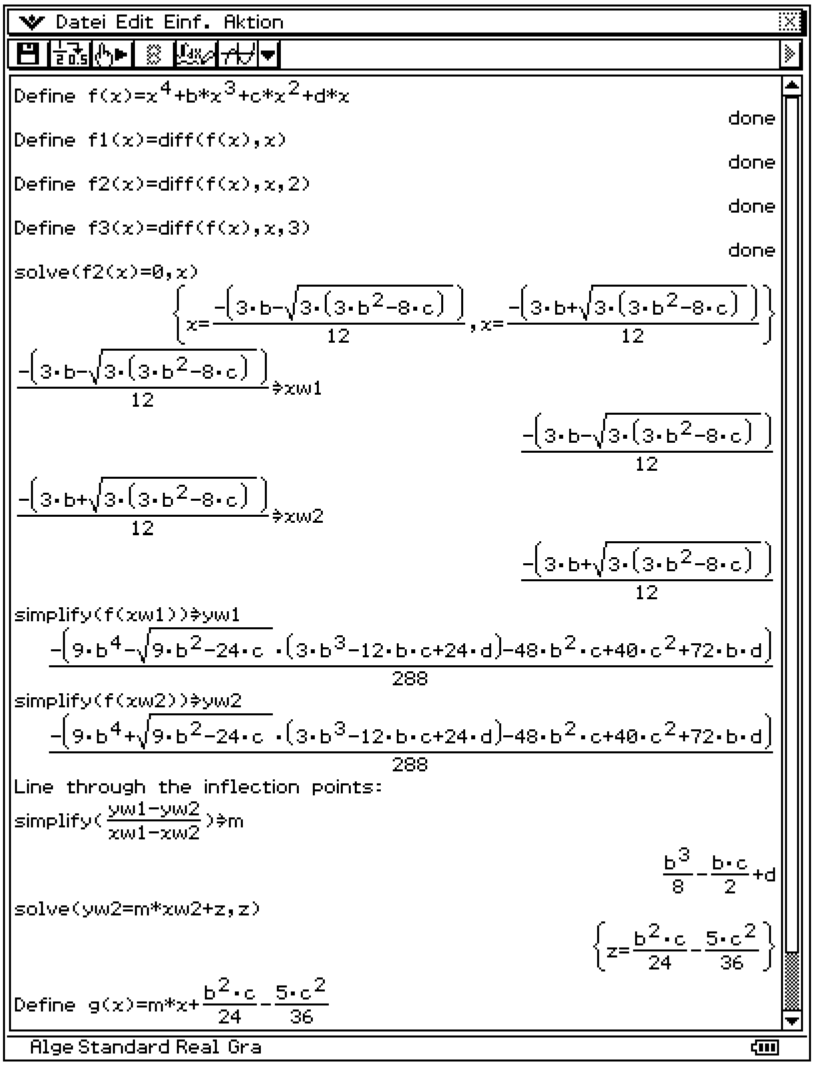
$$z = \frac{b^2 \cdot c}{24} - \frac{5 \cdot c^2}{36}$$

So the straight line g(x) can be defined as

$$g(x) = m \cdot x + (\frac{b^2 \cdot c}{24} - \frac{5 \cdot c^2}{36})$$

Intersection of the straight line g(x) with the function f(x):

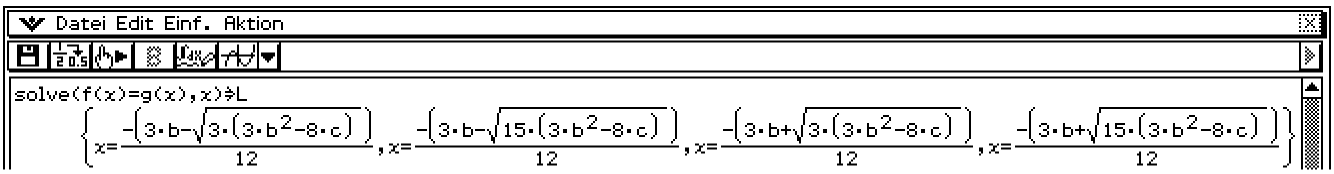
CAS: $solve(f(x) = g(x), x)$



$$x = xw1$$

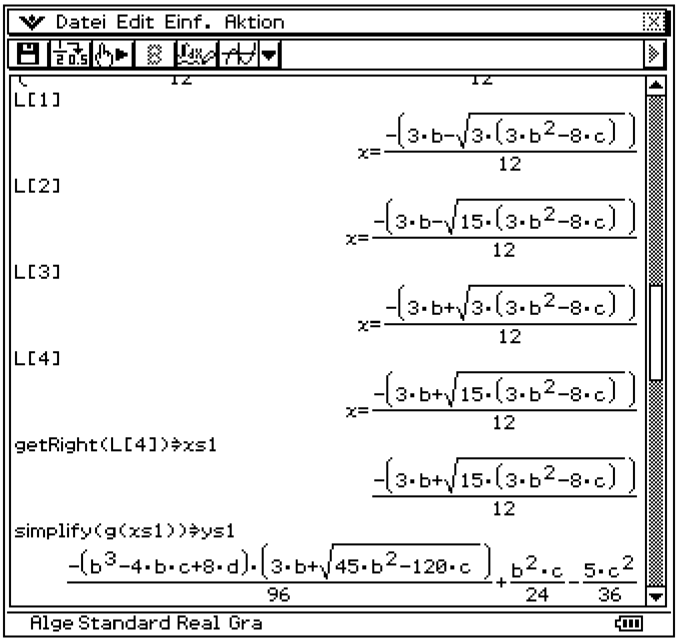
CAS delivers: $x = xw2$

$$x = \frac{-(3 \cdot b \pm \sqrt{15 \cdot (3 \cdot b^2 - 8 \cdot c)})}{12}$$



The last two solutions are the other intersection points. The one located furthest to the left is the negative one. It is stored as xs1.

The output g(xs1) is stored as ys1.



Distance between the two inflection points:

$$\sqrt{(yw2-yw1)^2+(xw2-xw1)^2} \Rightarrow dd$$

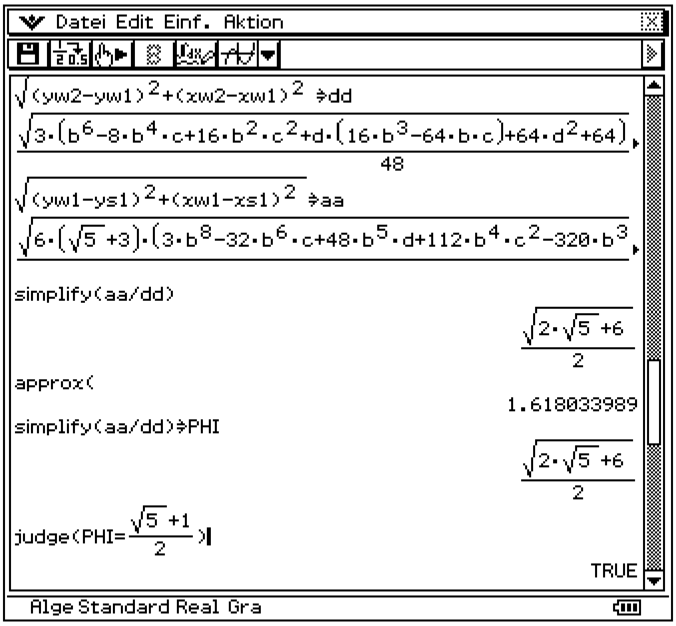
Distance between left intersection point of straight line with function and left inflection point:

$$\sqrt{(yw1-ys1)^2+(xw1-xs1)^2} \Rightarrow aa$$

After simplifying the term, the result the ClassPad delivers for the proportion of the segment between *aa* and *dd* is: $\dfrac{\sqrt{2\cdot\sqrt{5}+6}}{2}$

Approximation yields 1.618…

Using the "judge" command, we can confirm that the result equals $\Phi = \dfrac{\sqrt{5}+1}{2}$.



Following the same routine, we see that the proportion dd/aa equals

$$\varphi = \frac{\sqrt{5}-1}{2}.$$